



Meshless Methods

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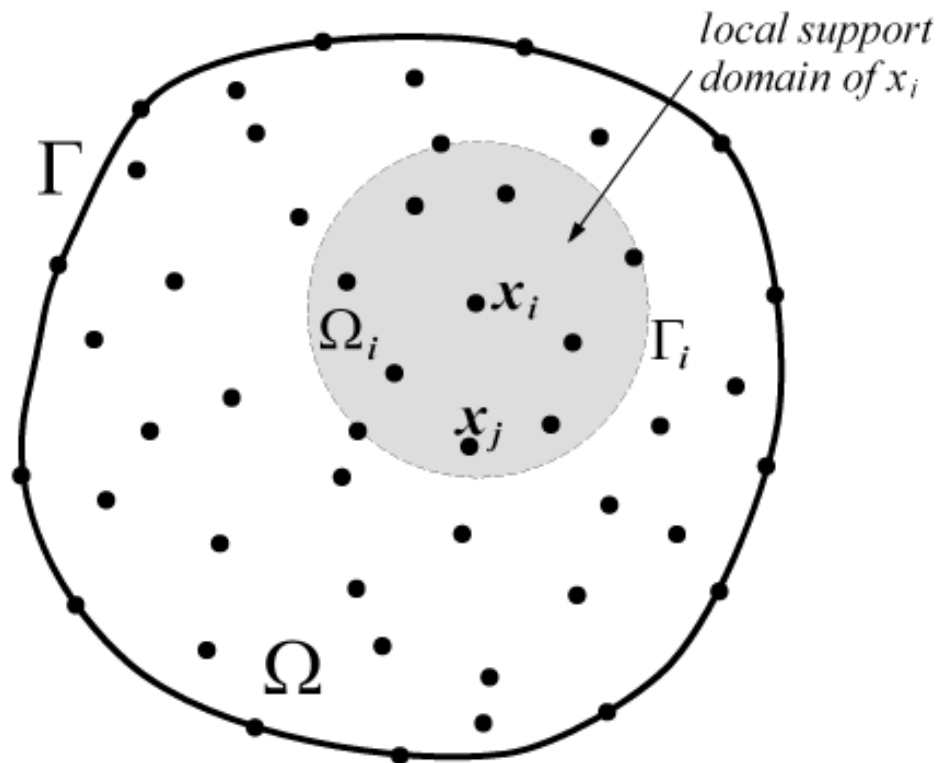
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Moving Least Squares (MLS):

- MLS provides a continuous and smooth approximation in the entire domain.
- MLS is capable of producing an approximation with the desired order of consistency.



$$j = 1, \dots, n_i$$

$$n_i > m$$

Least Square Approximation (LS):

- Assume:
 - $u(x)$ = the field variable in the domain
 - $u^h(x)$ = the approximation of $u(x)$ at point x
- The MLS approximation is defined as:

$$u^h(x) = p^T(x)a(x) = \sum_{j=1}^m p_j(x)a_j(x)$$

where

$$a^T(x) = \{a_0(x), a_1(x), \dots, a_m(x)\}$$

$$p^T(x) = \{p_0(x), p_1(x), \dots, p_m(x)\}$$

Moving Least Squares (MLS):

- Approximated values of the field function at the nodal points x_j can be defined as:

$$u^h(x, x_j) = P^T(x_j)a(x) \quad , j = 1, 2, \dots, n$$

- A functional of weighted residuals is constructed using the approximated values of the field function and the nodal parameters $u_j = u(x_j)$:

$$\Pi = \sum_{j=1}^{n_i} W(x - x_j) [u^h(x, x_j) - u_j]^2$$

$$\Pi = \sum_{j=1}^{n_i} W(x - x_j) [P^T(x_j)a(x) - u_j]^2$$

Moving Least Squares (MLS):

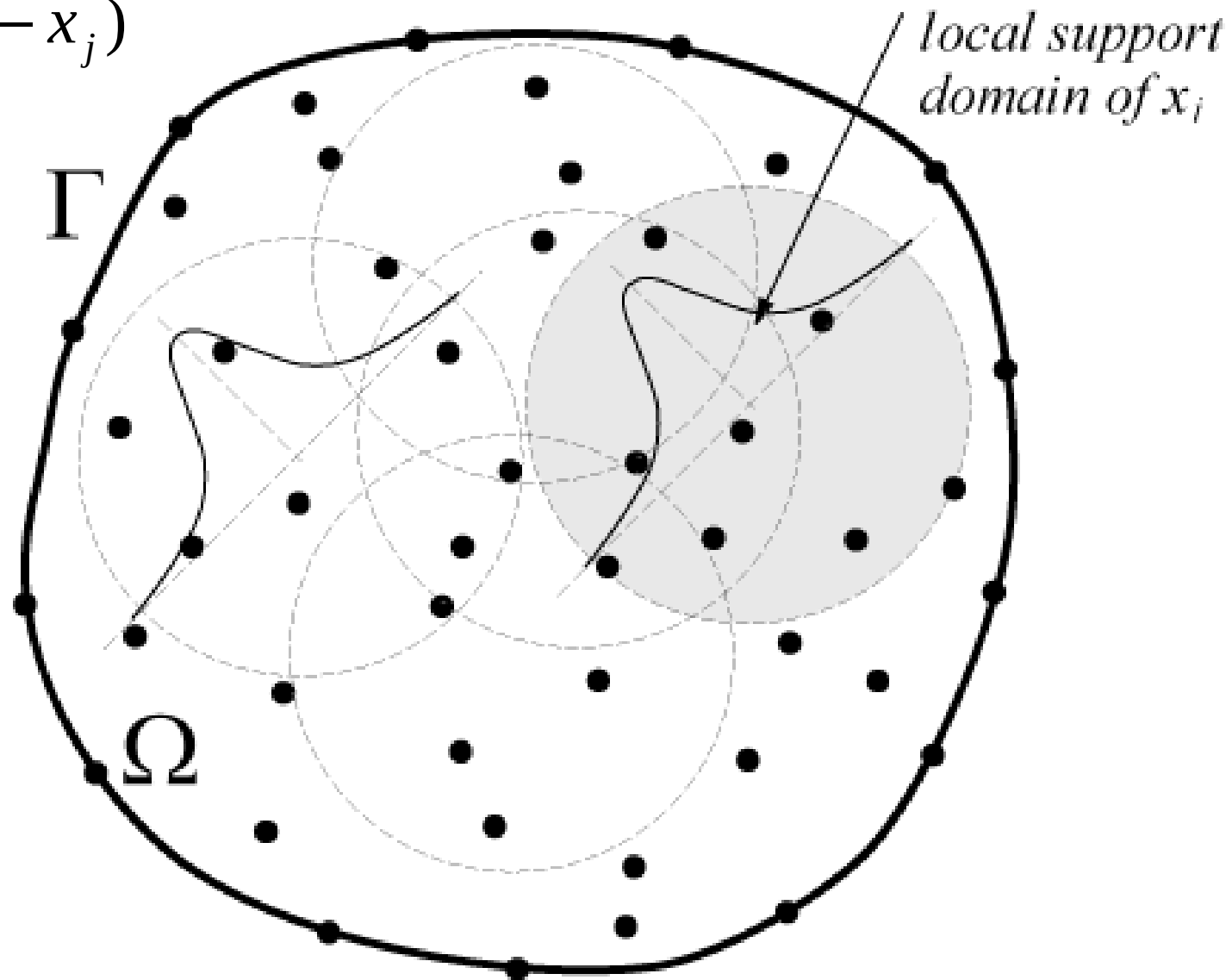
- The weight function W has two main features:
 - provides weightings for the residuals at different nodes in the support domain.
 - ensures that nodes leave or enter the support domain in a smooth (gradual) manner when x moves. As a result the compatibility condition is satisfied.
- The weight function is expressed as:

$$W(x - x_j)$$

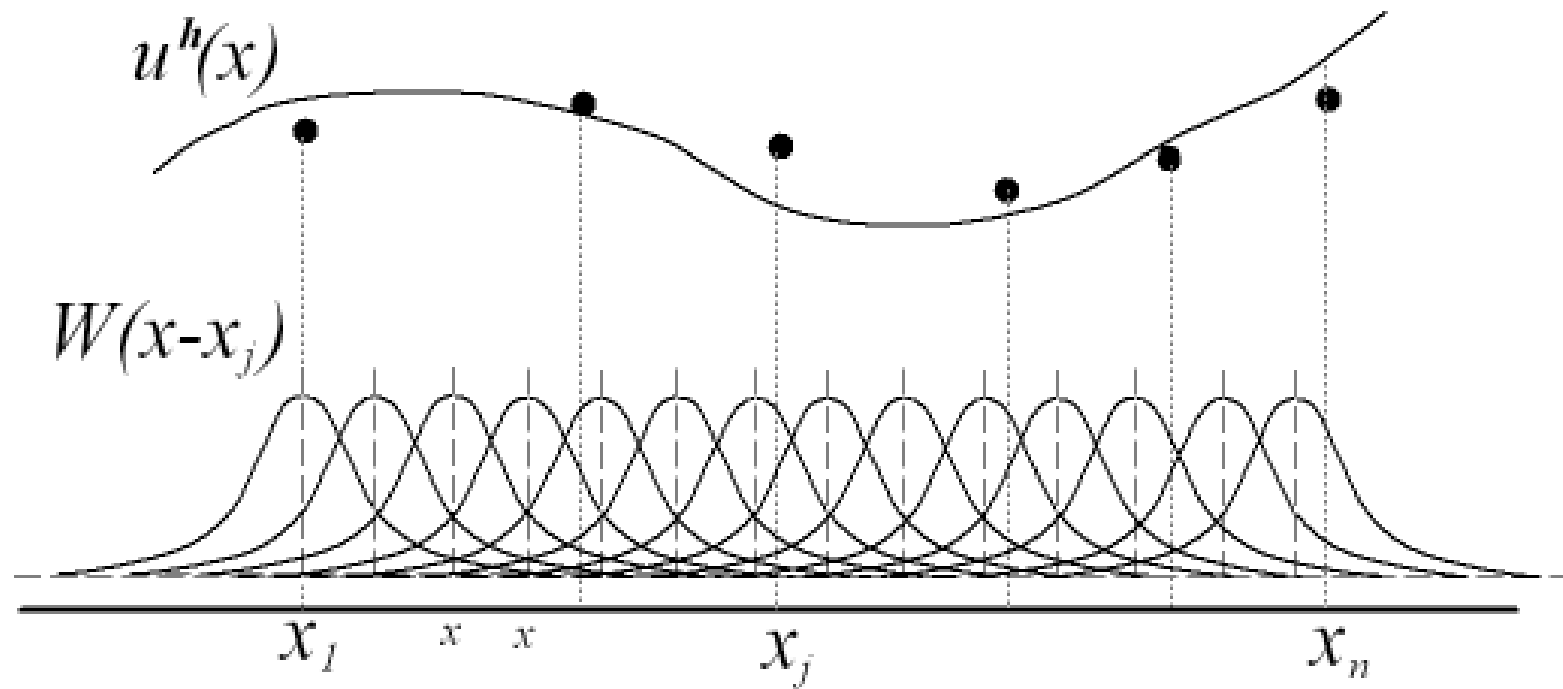
which illustrates that W is defined for each point (and not node); the 'moving' nature of the MLS.

Moving Least Squares (MLS):

$$W(x - x_j)$$



Moving Least Squares (MLS):



Moving Least Squares (MLS):

- At an arbitrary point x , $a(x)$ is chosen to minimize the weighted residual functional:

$$\frac{\partial \Pi}{\partial a} = 0$$

- Results in:

$$A(x)a(x) = B(x)U_s$$

- where

$$A(x) = \sum_{j=1}^{n_i} W(x - x_j) P(x_j) P^T(x_j)$$

$$B(x) = [W(x - x_1)P(x_1), \dots, W(x - x_{n_i})P(x_{n_i})]$$

$$U_s = [u_1, u_2, \dots, u_{n_i}]^T$$

Moving Least Squares (MLS):

- Solving $A(x)a(x) = B(x)U_s$ for $a(x)$

$$a(x) = A^{-1}(x)B(x)U_s$$

- Results in:

$$u^h(x, x_j) = P^T(x_j)A^{-1}(x)B(x)U_s$$

$$u^h(x, x_j) = \sum_{j=1}^{n_i} \sum_{k=1}^m P^T(x) \left[A^{-1}(x)B(x) \right]_{kj} u_j$$

Moving Least Squares (MLS):

- or in terms of the MLS shape functions $\phi(x)$

$$u^h(x) = \phi(x)U_s = \sum_{j=1}^{n_i} \phi_j(x)u_j$$

$$\phi_j(x) = P^T A^{-1} B_j = \sum_{k=1}^m P_k(x) [A^{-1}(x) B(x)]_{kj}$$

- The vector of MLS shape functions:

$$\phi(x) = [\phi_1(x), \phi_2(x), \dots, \phi_{n_i}(x)]$$

Moving Least Square (MLS):

- The MLS shape functions do not satisfy the Kronecker delta property:

$$\phi_j(x_l) \neq \delta_{jl}$$

- The nodal parameters u_j are not the same as the nodal values:

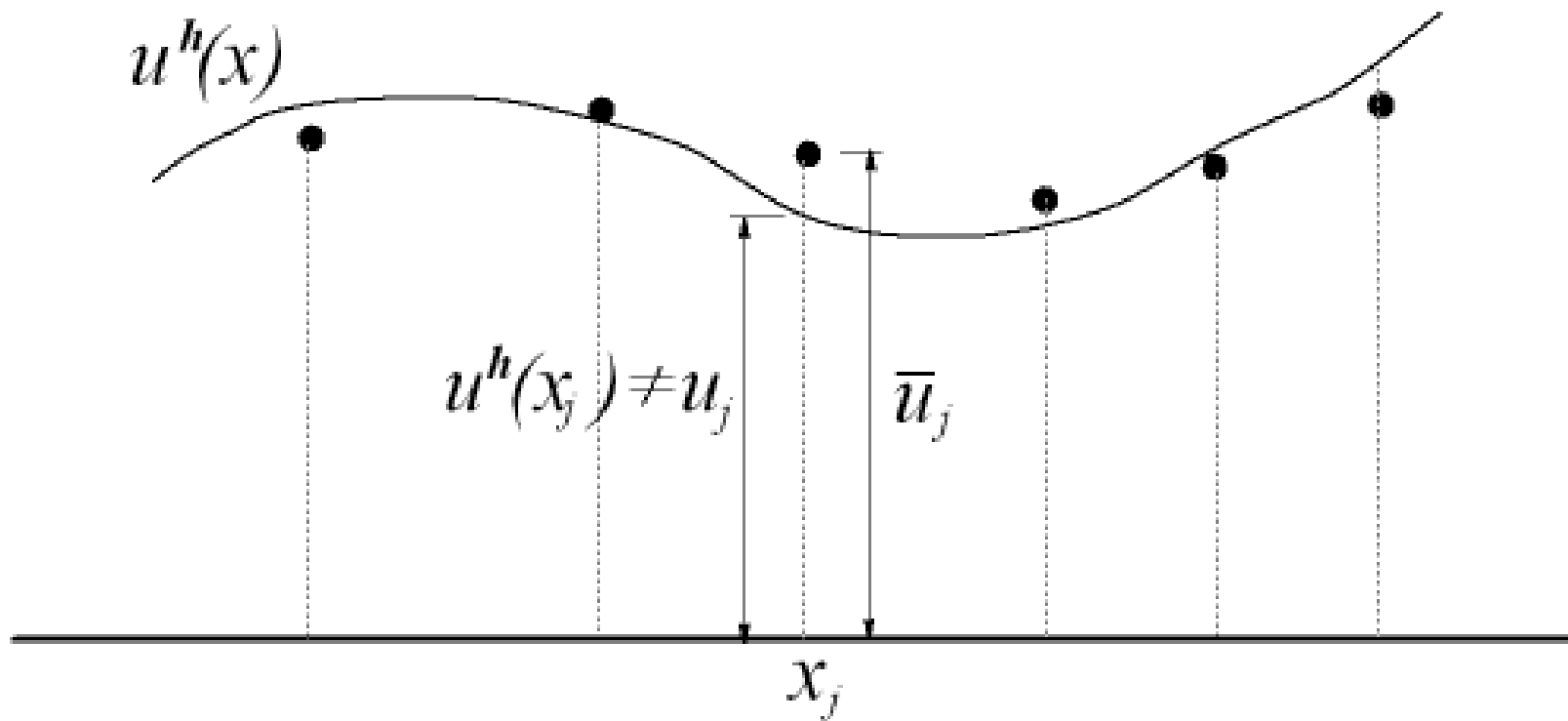
$$u^h(x_j) \neq u_j$$

As a result, the method is not an interpolation.

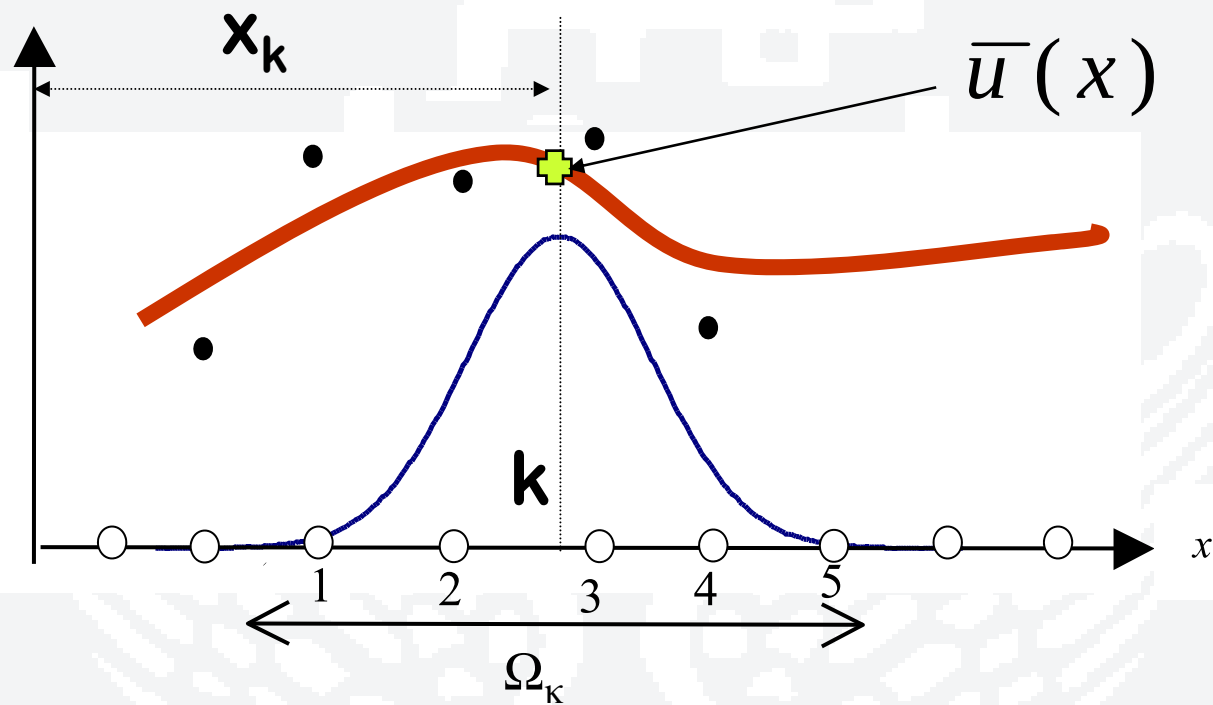
- Approximation of the displacement at a node, $u^h(x_j)$, depends to all the nodes within the support domain of that node ($j=1, n_j$). This property makes the imposition of essential boundary conditions more complicated than in FEM.

Moving Least Squares (MLS):

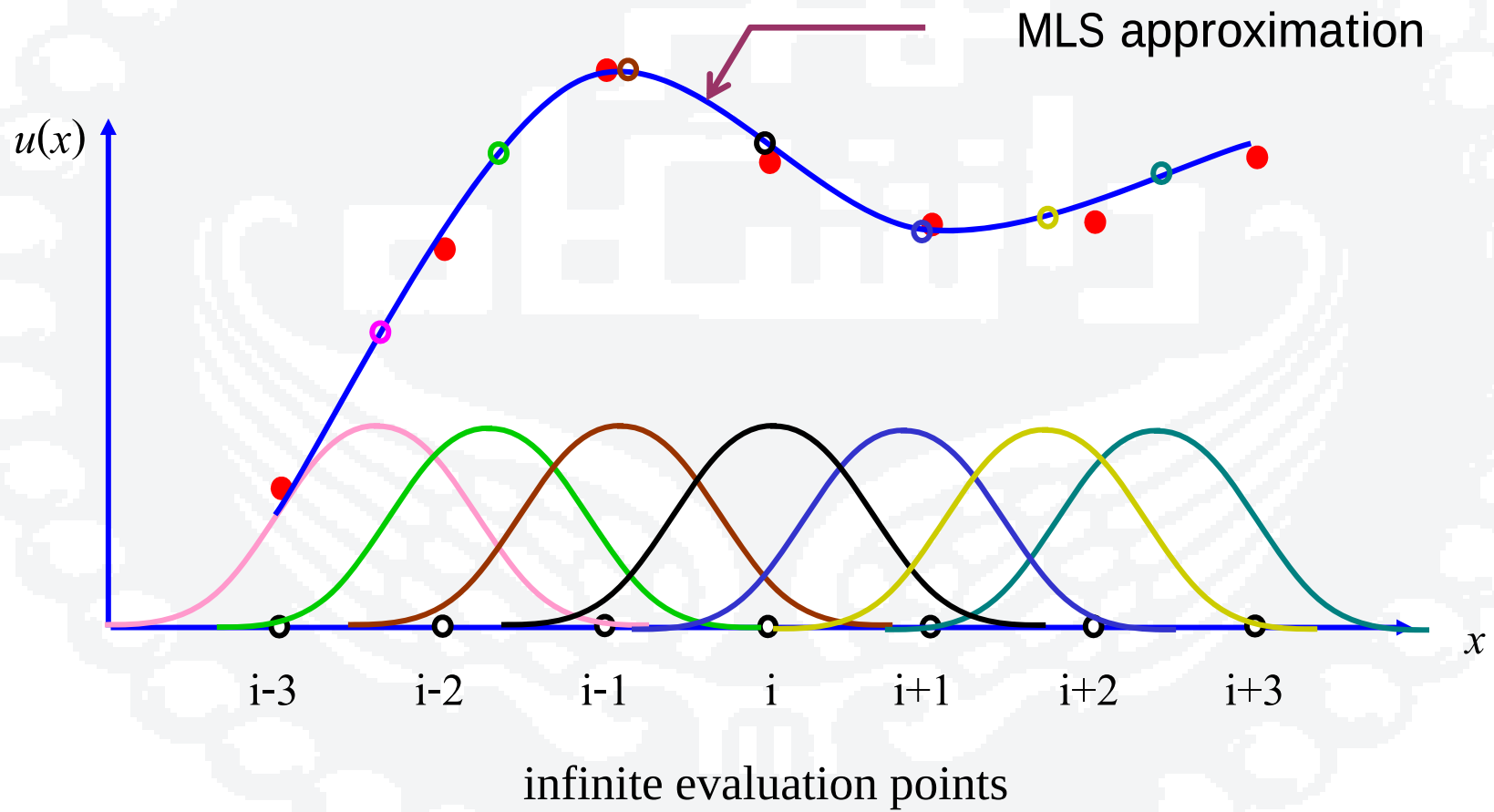
$$u^h(x_j) \neq u_j$$



Moving Least Squares (MLS):



Moving Least Squares (MLS):



Moving Least Square (MLS):

- In MLS approximation,

$$u^h(x) = \sum_{j=1}^{n_i} \phi_j(x) u_j$$

- u_j are nodal parameters found by setting up the interpolation condition at the nodal points x_m :

$$\bar{u}_m = u(x_m) = u^h(x_m) = \sum_{j=1}^{n_i} \phi_j(x_m) u_j$$

after computation of u_j , the MLS approximation may once again be used to evaluate the approximation at any point:

$$u^h(x) = \sum_{j=1}^{n_i} \phi_j(x) u_j$$

Moving Least Squares (MLS):

- The shape functions can be constructed with arbitrary level of continuity.
- The boundaries of the support domain do not largely affect the smoothness of the shape functions.
- The smoothness of the MLS shape function is given by:
 - The smoothness of the weight function
 - The order of the basis function
- The arbitrary overlaps of the nodal support domains lead to a variable order of connectivity.
- MLS shape functions are non-polynomial.
- A different order of approximation is obtained for each point.

Moving Least Squares (MLS):

Summary:

$$u^h(x) = \sum_{k=1}^m p_k(x) a_k(x) = \{p(x)\}_{1 \times m}^T \{a(x)\}_{m \times 1}$$

$$\Pi(x) = \frac{1}{2} \sum_{j=1}^n W(x - x_j) \left[\sum_{k=1}^m p_k(x_j) a_k(x_j) - u_j \right]^2$$

$$\Pi(x) = \frac{1}{2} \{e\}_{n \times 1}^T [W(x)]_{n \times n} \{e\}_{n \times 1}$$

$$\{e\}_{n \times 1} = [P]_{n \times m} \{a(x)\}_{m \times 1} - \{u\}_{n \times 1}$$

Moving Least Squares (MLS):

Summary:

$$\{u\}^T = \{u_1, u_2, \dots, u_{n_i}\}_{1 \times n}$$

$$[P]_{n \times m} = \begin{bmatrix} p_1(x_1) & p_2(x_1) & \cdots & p_m(x_1) \\ p_1(x_2) & p_2(x_2) & \cdots & p_m(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(x_n) & p_2(x_n) & \cdots & p_m(x_n) \end{bmatrix}_{n \times m}$$

$$[W(x)]_{n \times n} = \begin{bmatrix} W(x-x_1) & 0 & \cdots & 0 \\ 0 & W(x-x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W(x-x_n) \end{bmatrix}_{n \times n}$$

Moving Least Squares (MLS):

Summary:

$$\left\{ \frac{\partial \Pi}{\partial a} \right\}_{m \times 1} = [A(x)]_{m \times m} \{a(x)\}_{m \times 1} - [B(x)]_{m \times n} \{u\}_{n \times 1} = \{0\}_{m \times 1}$$

$$[A(x)]_{m \times m} = [P]_{m \times n}^T [W(x)]_{n \times n} [P]_{n \times m}$$

$$[B(x)]_{m \times n} = [P]_{m \times n}^T [W(x)]_{n \times n}$$

$$\{a(x)\}_{m \times 1} = [A^{-1}(x)]_{m \times m} [B(x)]_{m \times n} \{u\}_{n \times 1}$$

Moving Least Squares (MLS):

Summary:

$$u^h(x) = \sum_{j=1}^n \phi_j(x) u_j = \{\phi(x)\}_{1 \times n}^T \{u\}_{n \times 1}$$

$$\{\phi(x)\}_{1 \times n}^T = [\phi_1(x), \phi_2(x), \dots, \phi_n(x)]$$

$$\{\phi(x)\}_{1 \times n}^T = \{p(x)\}_{1 \times m}^T [A^{-1}(x)]_{m \times m} [B(x)]_{m \times n}$$

Moving Least Squares (MLS):

- MLS weight functions should be defined on a compact support, ensuring the sparsity of the global matrices.
- Arbitrary weight functions can not act as shape functions, because the consistency conditions may not be satisfied.
- The weight functions should be non-negative.
- At least m weight functions should be non-zero at each x .
- The weight functions should be continuous up to the k^{th} derivative if a continuous k^{th} derivative of the shape function is desired.

Moving Least Squares (MLS):

The moment matrix A :

- A is symmetric.
- In order to obtain a unique solution for $a(x)$, $A(x)$ must be non-singular.
- A may become ill-conditioned when:
 - The basis functions are (almost) linearly dependent.
 - There are not enough nodal supports overlapping the given point.
 - The nodes whose supports overlap at a point are arranged in a special pattern.
- If the components of $p(x)$ are independent, then $A(x)$ is non-singular if at least m weight functions are non-zero at a certain location of evaluation x_i .

Moving Least Squares (MLS):

The moment matrix A :

- Possible checks for the precision of the solutions of A :
 - Indirect check: estimate the condition number of A
 - Direct check: check the final accuracy of the shape functions by invoking the first consistency condition:

$$\sum_{j=1}^n \phi_j(x) = 1$$

- Check the accuracy of derivatives

$$\sum_{j=1}^n \phi_{j,i}(x) = 0 \quad , i = x, y, z$$

Moving Least Squares (MLS):

The basis function $p(x)$:

- Linear

$$p^T(x) = \{1, x\}$$

- Quadratic

$$p^T(x) = \{1, x, x^2\}$$

- Radial enrichment

$$p^T(x) = \{1, x, y, \sqrt{r}\}$$

- Full enrichment

$$p^T(x) = \left\{1, x, y, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \frac{\theta}{2}, \dots\right\}$$

- Fourier based

$$p^T(x) = \{1, \cos kx, \sin kx, \dots\}$$

Moving Least Squares (MLS):

Derivatives of the MLS shape functions

$$\phi_i(x) = P^T A^{-1} B_i = \sum_{k=1}^m P_k(x) [A^{-1}(x) B(x)]_{ki}$$

$$\phi_{i,x}(x) = \left(P^T A^{-1} B_i \right)_{,x} = P_{,x}^T A^{-1} B_i + P^T \left(A^{-1} \right)_{,x} B_i + P^T A^{-1} B_{i,x}$$

$$B_{i,x} = \frac{dW(x-x_i)}{dx} p(x_i)$$

$$\left(A^{-1} \right)_{,x} = -A^{-1} A_{,x} A^{-1}$$

$$A_{,x} = \sum_{j=1}^n \frac{dW(x-x_j)}{dx} p(x_j) p^T(x_j)$$

$$= \frac{dW(x-x_1)}{dx} \begin{bmatrix} 1 & x_1 \\ x_1 & x_1^2 \end{bmatrix} + \dots + \frac{dW(x-x_n)}{dx} \begin{bmatrix} 1 & x_n \\ x_n & x_n^2 \end{bmatrix}$$

Moving Least Squares (MLS):

Derivatives of the MLS shape functions

$$u^h(x) = \Phi(x)U_s$$

$$\Phi(x) = \gamma^T(x)B(x)$$

$$A(x)\gamma(x) = B(x)$$

$$\Phi_{,i} = \gamma_{,i}B + \gamma B_{,i}$$

$$\Phi_{,ij} = \gamma_{,ij}B + \gamma_{,i}B_{,j} + \gamma_{,j}B_{,i} + \gamma B_{,ij}$$

$$\begin{aligned}\Phi_{,ijk} = & \gamma_{,ijk}B + \gamma_{,ij}B_{,k} + \gamma_{,ik}B_{,j} + \gamma_{,i}B_{,jk} \\ & + \gamma_{,jk}B_{,i} + \gamma_{,j}B_{,ik} + \gamma_{,k}B_{,ij} + \gamma B_{,ijk}\end{aligned}$$

Moving Least Squares (MLS):

Derivatives of the MLS shape functions

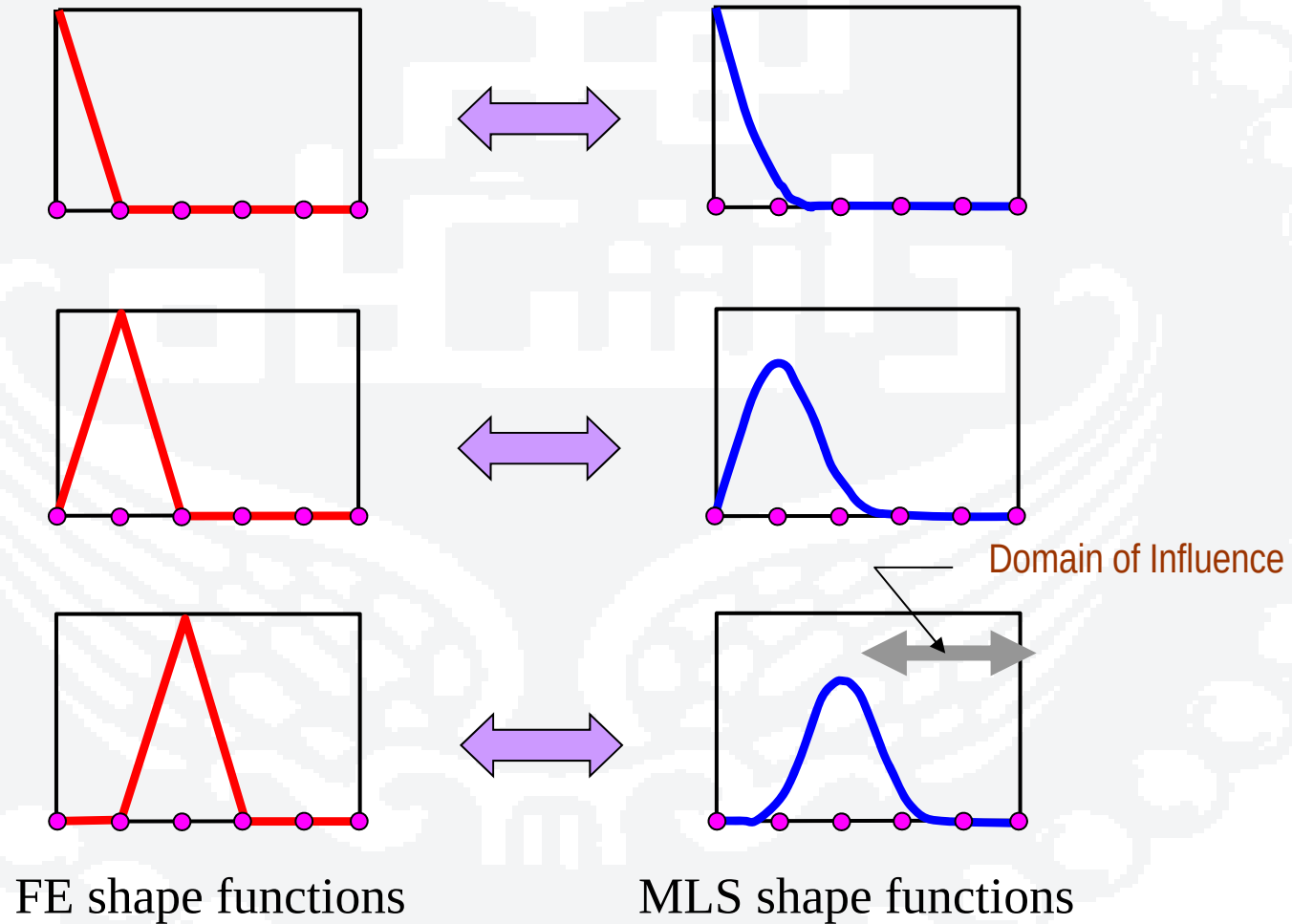
$$A(x)\gamma(x) = B(x)$$

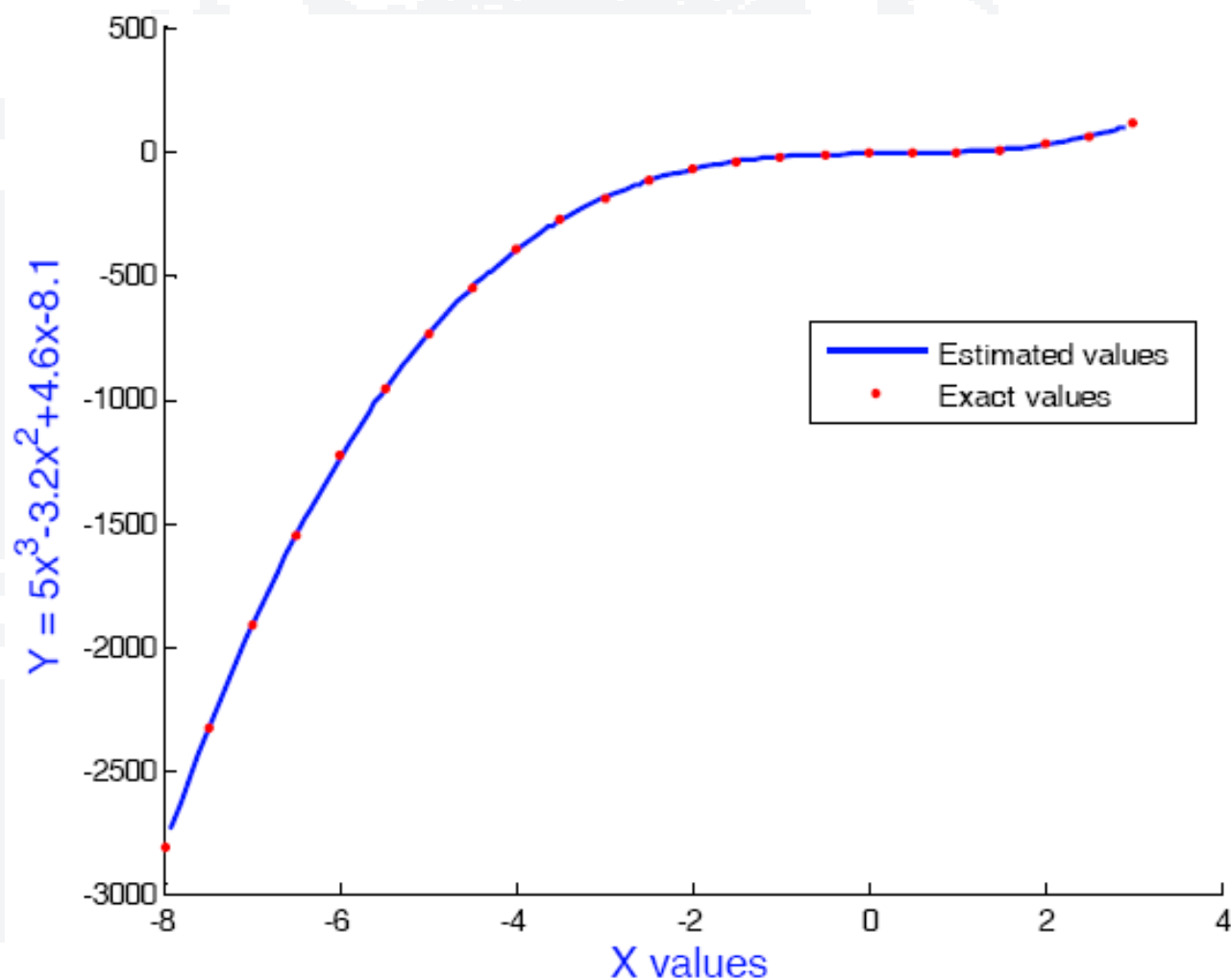
$$A\gamma_{,i} = P_{,i} - A_{,i}\gamma$$

$$A\gamma_{,ij} = P_{,ij} - (A_{,i}\gamma_{,j} + A_{,j}\gamma_{,i} + A_{,ij}\gamma)$$

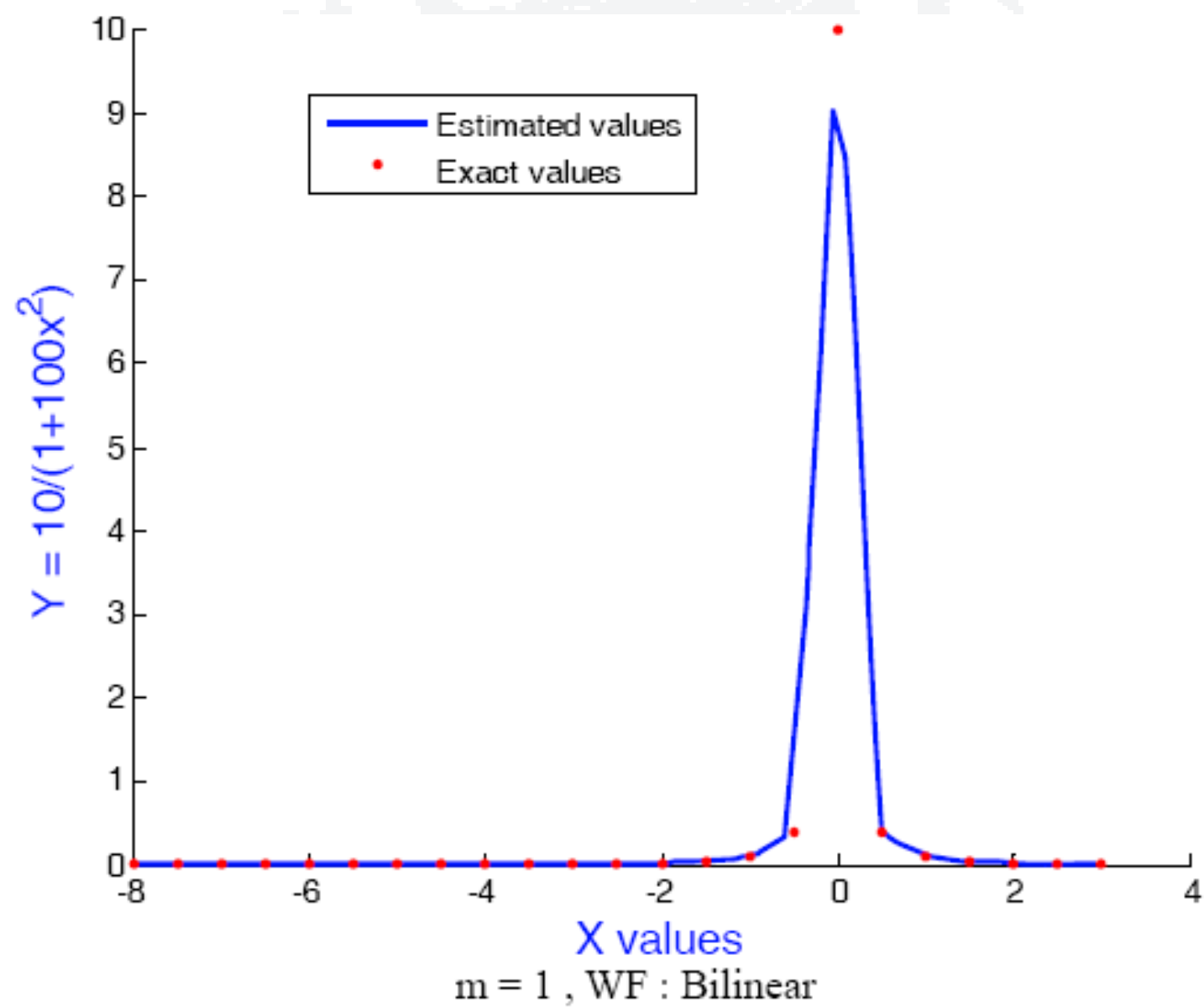
$$\begin{aligned} A\gamma_{,ijk} = P_{,ijk} - (A_{,i}\gamma_{,jk} + A_{,j}\gamma_{,ik} + A_{,k}\gamma_{,ij} + A_{,ij}\gamma_{,k} \\ + A_{,ik}\gamma_{,j} + A_{,ij}\gamma_{,i} + A_{,ijk}\gamma) \end{aligned}$$

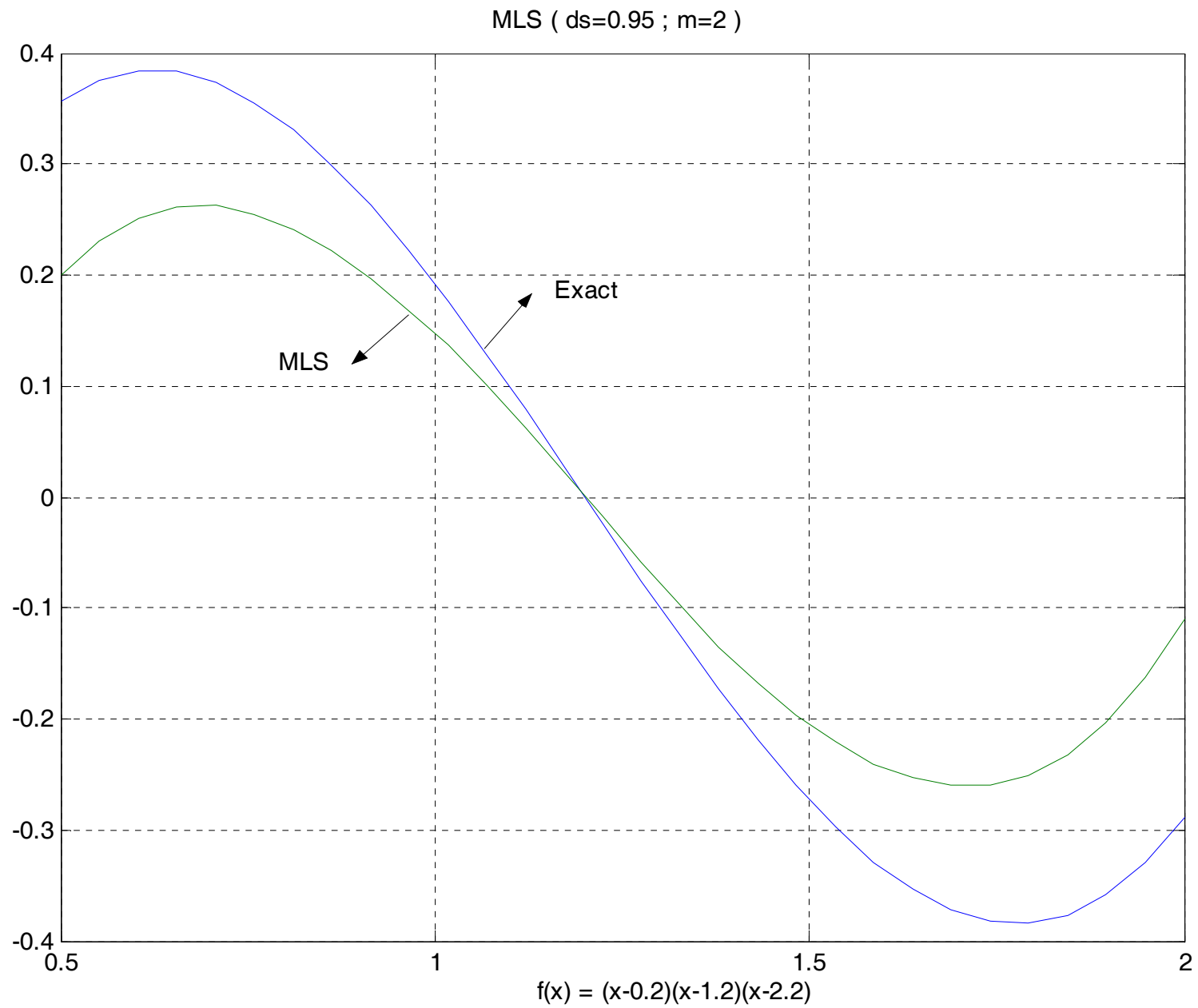
Moving Least Squares (MLS):

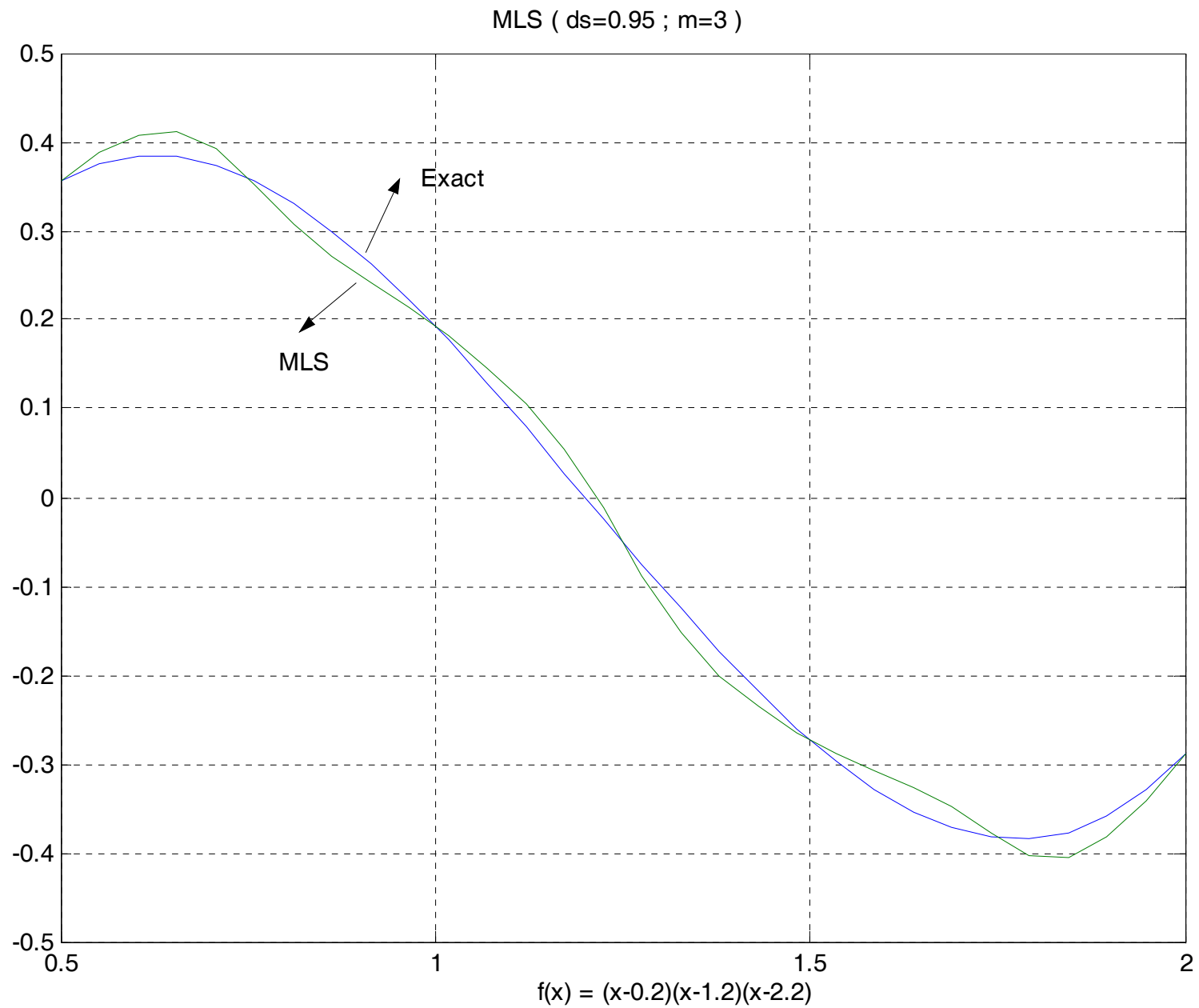




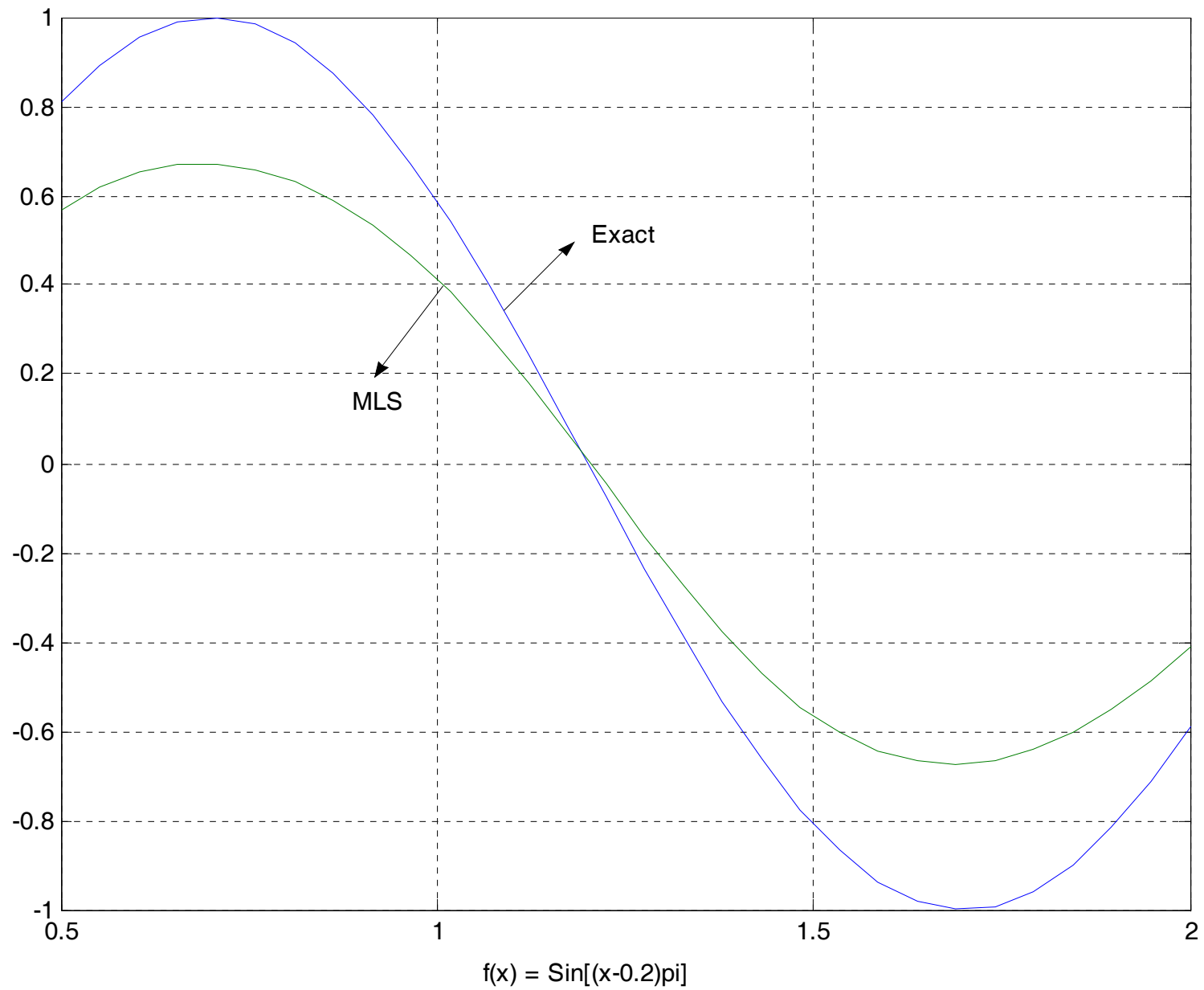
m = 5 , WF : Cubic
Mean squared error = 0.0065

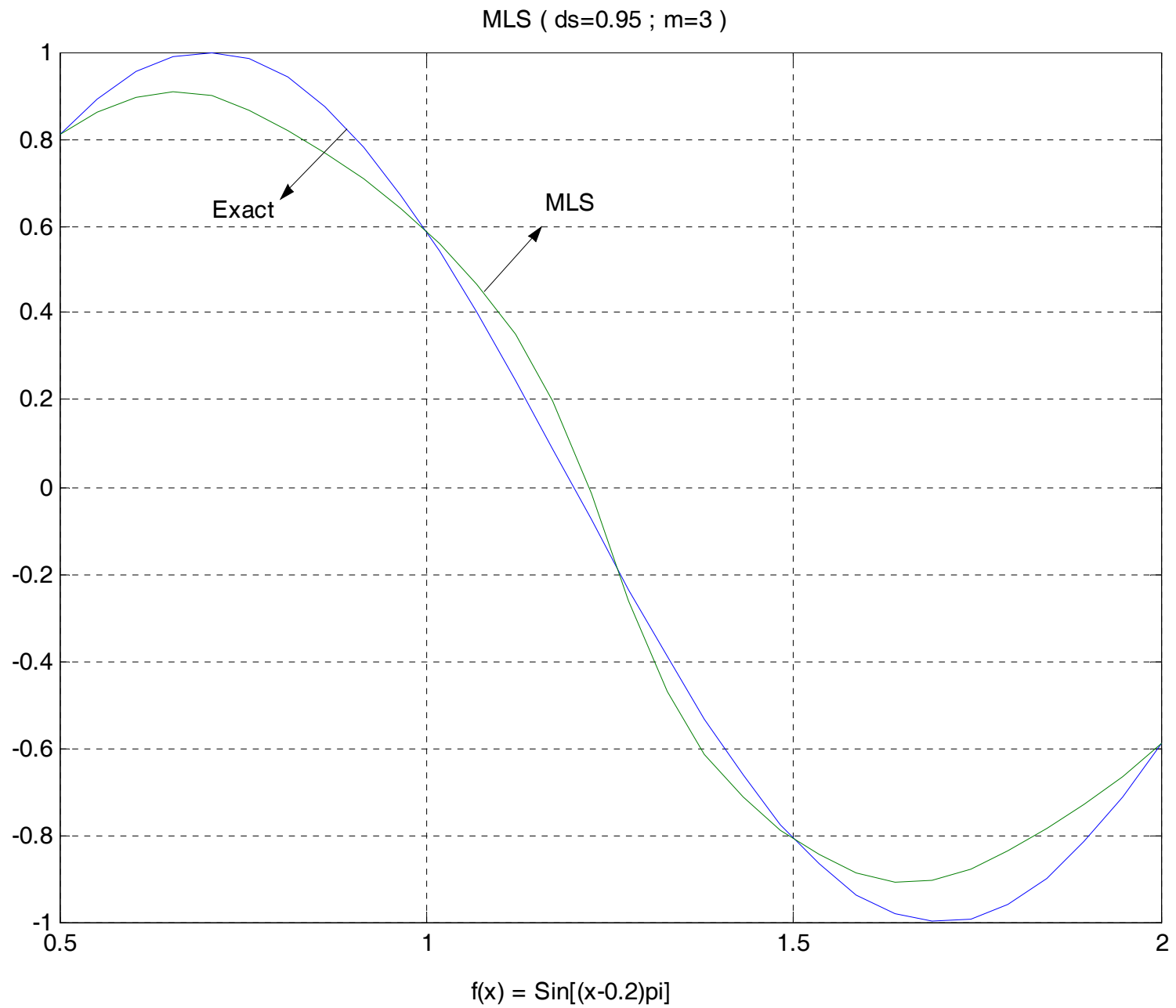




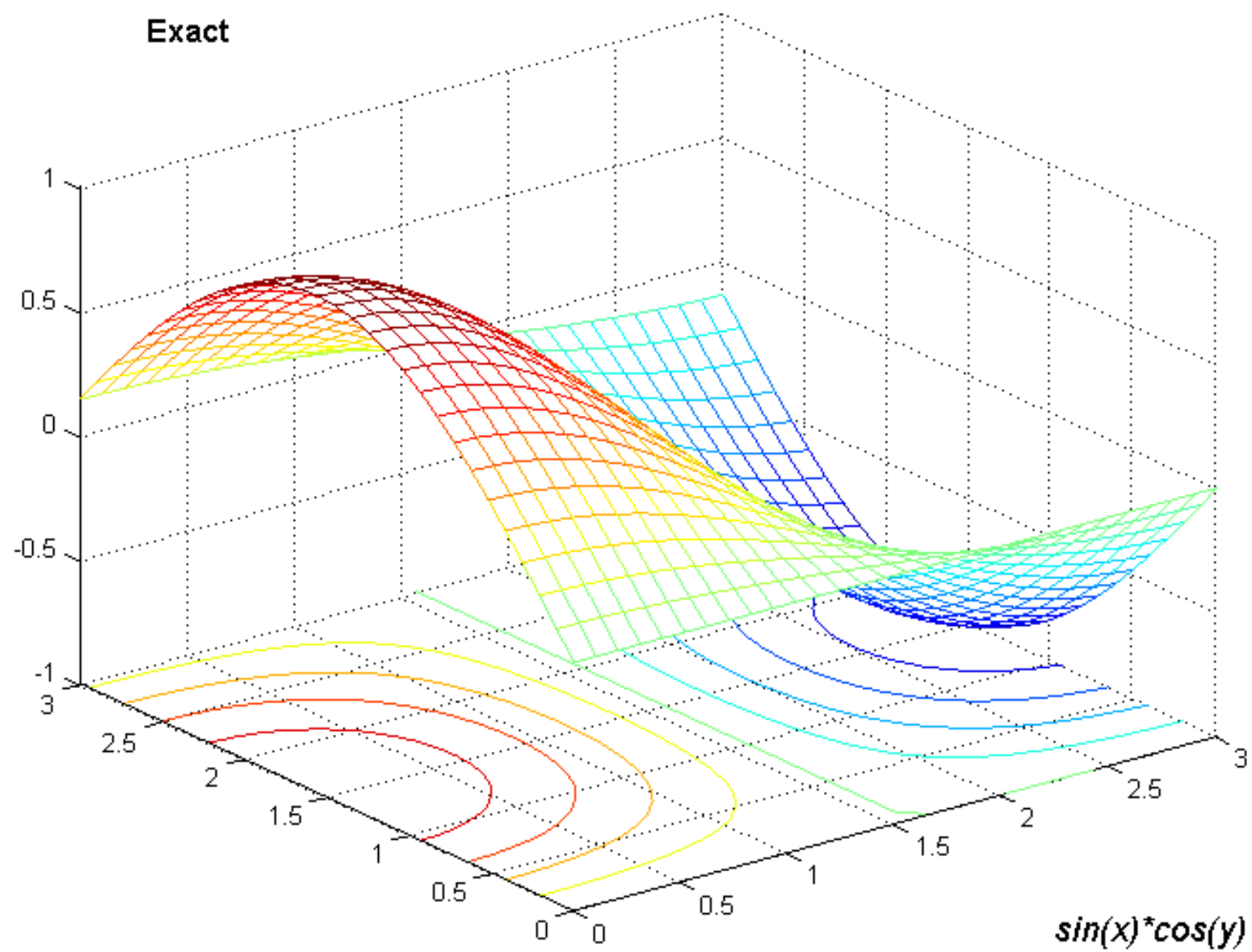


MLS (ds=0.95 ; m=2)

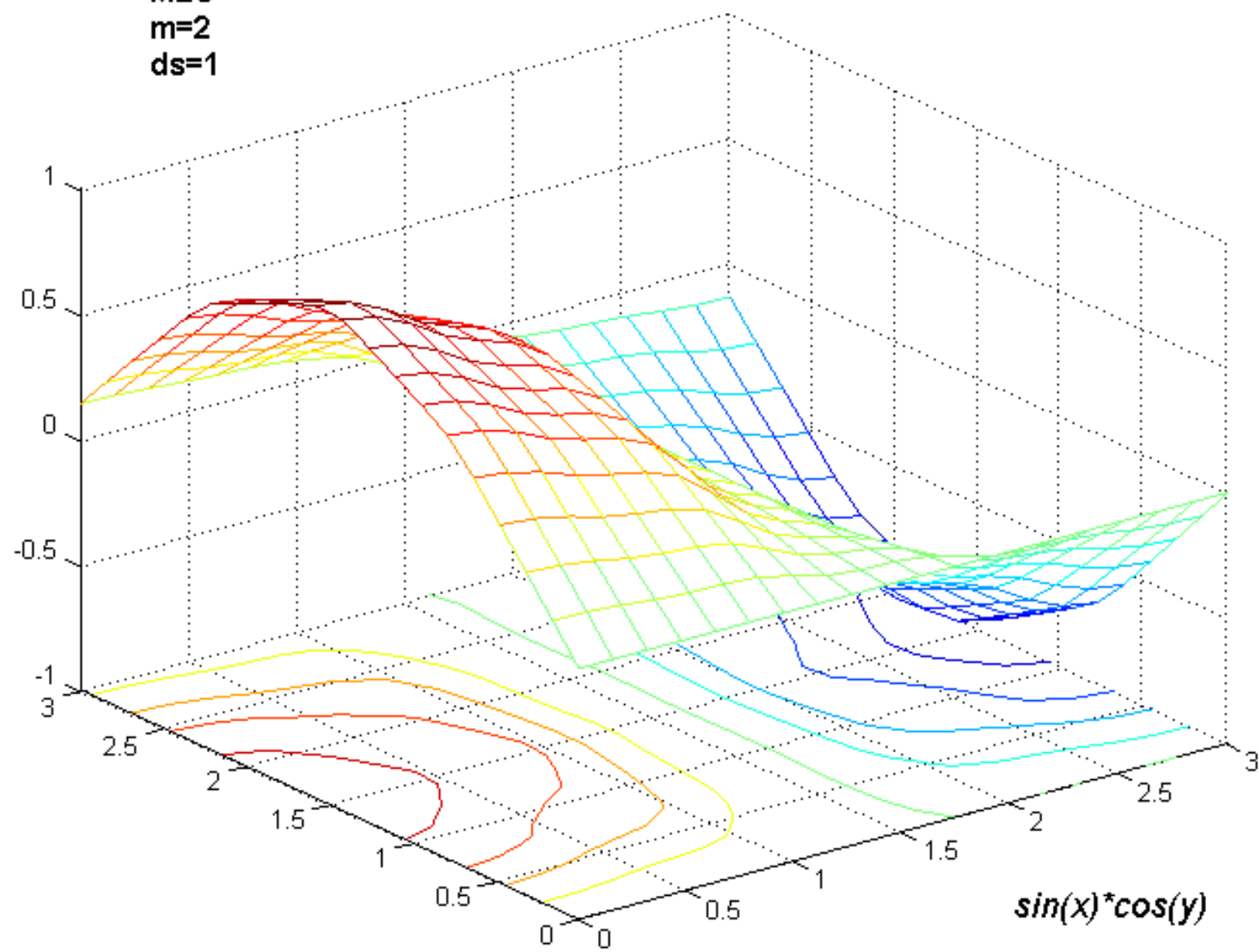




Exact



MLS
m=2
ds=1



MLS
m=3
ds=1.5

